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# WRAPPABILITY OF CURVES ON SURFACES

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US ARMY ARMAMENT RESEARCH,
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# WRAPPABILITY OF CURVES ON SURFACES

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ABSTRACT. In this paper, conditions are derived under which a path on a general smooth surface is wrappable or capable of receiving an essentially one-dimensional flexible filament under tension that clings to the surface throughout its length and does not slip. Wrappability considerations are of practical importance in the fabrication of filament-wound composite pressure vessels, for instance. The general wrappability conditions derived are applied to two special cases: general cylinders and general surfaces of revolution.

INTRODUCTION. Imagine a rotating spindle accepting string from some delivery point which moves parallel to the axis of the spindle. This is the essence of the filament winding process where the "string" is replaced by a band of epoxy impregnated fiber glass, for instance; layer upon layer of this filament is evenly laid down; and the whole thing is ultimately cured or baked into a filament-wound composite structure - often a pressure vessel. The spindle or mandrel is designed so that it may be broken down into parts and removed subsequent to curing, leaving only the wrappings embedded in the matrix material. This paper considers the question of how the winding or wrapping process is limited by the differential geometric nature of the mandrel's surface.

PRELIMINARIES. Begin with point P in three space:

$$P = iX + jY + kZ$$

Restrict P by parameterizing with respect to x and $\theta$, defining a surface S embedded in three space:

$$X = x$$

$$Y = r(x,\theta)\sin\theta$$

$$Z = r(x,\theta)\cos\theta$$

where $r(x,\theta)$ is the radius of the surface. We require r to be sufficiently smooth, positive, and $2\pi$ periodic in $\theta$, making S a closed surface with an inside and an outside. If P is further restricted by defining $\theta$ in terms of x, P will lie on a curve or path embedded in the surface S.

Let $(\ )'$ denote $\frac{d}{dx}(\ )$, and let s denote distance along curve c. The tangent vector t to curve c is

$$t = \frac{dP}{ds} = P'(s')^{-1}$$

and the curvature vector $\kappa$ of any curve c is

$$\kappa = \frac{dt}{ds} = t'(s')^{-1}$$

$$= (P''-ts'')(s')^{-2}$$

A family of vectors tangent to surface S at point P is

$$dP = P_X dx + P_\theta d\theta = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial \theta} d\theta$$

Two independent vectors spanning the tangent space at P are therefore $P_X$ and $P_\theta$. Now form the vector cross product of $P_X$ and $P_\theta$ to obtain $\nu$, a vector normal to the surface and pointing away from the outside of the surface.

$$\nu = P_X \times P_\theta$$

WRAPPABILITY CONDITION I - NO LIFTOFF. Now, in order for c to be a wrappable curve on surface S, it is necessary for a length of flexible filament under tension to cling to c and S. In order for this clinging to take place, it is necessary for c to see the outside of S as being convex. This will be the case only if the curvature vector of c points away from the inside of the surface. It is therefore necessary that the inner or dot product of $\nu$ and $\kappa$ be negative for clinging to take place. Now,

$$\nu \cdot \kappa = \nu \cdot (P''-ts'')(s')^{-2}$$

but

$$\nu \cdot t = 0$$

therefore,

$$\nu \cdot \kappa = \nu \cdot P''(s')^{-2}$$

and since only the sign of $\nu \cdot \kappa$ matters here, the function $\lambda$ is defined as

$$\lambda = \nu \cdot P''$$

When $\lambda$ is positive, a filament under tension tends to lift off the surface and form a bridge between two distant points; when $\lambda$ is negative, the filament tends to cling to the surface.

Now, the evaluation of $\lambda$ in terms of x, $\theta$, and r is outlined.

First,

$$P' = P_x + P_\theta \theta'$$

and

$$P'' = P_{xx} + 2P_{x\theta}\theta' + P_{\theta\theta}\theta'^2 + P_\theta \theta''$$

but $P_\theta$ is in the tangent space, so

$$\nu \cdot P_\theta = 0$$

Hence,

$$\lambda = \nu \cdot P'' = \nu \cdot P_{xx} + 2\nu \cdot P_{x\theta}\theta' + \nu \cdot P_{\theta\theta}\theta'^2$$

Note that all the inner products defining $\lambda$ are determined at a point on the surface independently of the curve c and that the only thing that changes $\lambda$ at a point is the direction of c determined by $\theta'$. Therefore, all curves with the same direction through a given point on the surface have the same value of $\lambda$ at that point.

Continuing the evaluation of $\nu \cdot P''$ we have

$$P_x = i + jY_x + kZ_x$$

$$P_\theta = jY_\theta + kZ_\theta$$

$$\nu = P_x x P_\theta = i(Y_x Z_\theta - Y_\theta Z_x) - jZ_\theta + kY_\theta$$

$$P_{xx} = jY_{xx} + kZ_{xx}$$

$$P_{x\theta} = jY_{x\theta} + kZ_{x\theta}$$

$$P_{\theta\theta} = jY_{\theta\theta} + kZ_{\theta\theta}$$

The dot products in $\lambda$ are therefore

$$\nu \cdot P_{xx} = Y_\theta Z_{xx} - Z_\theta Y_{xx}$$

$$\nu \cdot P_{x\theta} = Y_\theta Z_{x\theta} - Z_\theta Y_{x\theta}$$

$$\nu \cdot P_{\theta\theta} = Y_\theta Z_{\theta\theta} - Z_\theta Y_{\theta\theta}$$

Obtaining the partials in these dot products

$$Y = r \sin \theta$$

$$Y_x = Yr_x/r$$

$$Y_\theta = Yr_\theta/r + Z$$

$$Y_{xx} = Yr_{xx}/r$$

$$Y_{x\theta} = (Yr_{x\theta}+Zr_x)/r$$

$$Y_{\theta\theta} = \{Y(r_{\theta\theta}-r) + 2Zr_\theta\}/r$$

$$Z = r \cos \theta$$

$$Z_x = Zr_x/r$$

$$Z_\theta = Zr_\theta/r - Y$$

$$Z_{xx} = Zr_{xx}/r$$

$$Z_{x\theta} = (Zr_{x\theta}-Yr_x)/r$$

$$Z_{\theta\theta} = \{Z(r_{\theta\theta}-r) - 2Yr_\theta\}/r$$

The dot products then become

$$\nu \cdot P_{xx} = rr_{xx}$$

$$\nu \cdot P_{x\theta} = rr_{x\theta} - r_x r_\theta$$

$$\nu \cdot P_{\theta\theta} = rr_{\theta\theta} - 2r_\theta^2 - r^2$$

Hence,

$$\lambda = rr_{xx} + 2(rr_{x\theta}-r_x r_\theta)\theta' + (rr_{\theta\theta}-2r_\theta^2-r^2)\theta'^2$$

$$= a\theta'^2 + 2b\theta' + c$$

$$= a(\theta' + \frac{b}{a})^2 + c - \frac{b^2}{a}$$

It is clear that the sign of $\lambda$ is completely independent of $\theta'$ at points for which $ac > b^2$. Hence, any curve is wrappable where $ac > b^2$ and $a < 0$, but the surface is unwrappable if $ac > b^2$ and $a > 0$ anywhere. If $ac < b^2$, some curves will be wrappable and others won't. In any case, given the radius function r, its partial derivatives at a point, and the direction of a curve through that point, one can immediately compute whether or not a taut filament following the curve will tend to lift from the surface.

Consider two special cases. For a surface of revolution, $r_\theta = 0$. Therefore, $\lambda = rr'' - (r\theta')^2$ [1]. If $r'' < 0$ everywhere, all curves on the

surface of revolution are wrappable. Points for which $r'' > 0$ are also wrappable for curves with $\theta'$ sufficiently large. No surface of revolution is unwrappable in the sense of liftoff.

For a cylinder, $r_x = 0$, and

$$\lambda = a\theta'^2 = (rr_{\theta\theta}-2r_\theta^2-r^2)\theta'^2$$

Hence, every curve on a cylinder is wrappable if $a < 0$ everywhere, but if $a > 0$ anywhere, the cylinder is unwrappable.

In this section the phrase "is wrappable" has meant "does not experience filament liftoff or bridging during winding." In the next section, the definition of "wrappable" is augmented by considering friction between filament and surface.

WRAPPABILITY CONDITION II - NO SLIPPAGE. If there were no friction between filament and surface (or between filament and filament since the surface is filament after the first layer is laid down), there would be only one type of path along which one might wind filament without the filament slipping - a path with no transverse (tangent to the surface and perpendicular to the filament) forces acting on the filament - a path which curves neither left nor right in the surface - a path with zero geodesic curvature - a geodesic path. If there were no friction available, and only geodesic paths could be wound, there would be no filament winding industry. In fact, for numerous reasons, it is seldom if ever possible to wind along geodesic paths in practice [1]. The geodesic path remains as an ideal, however, and in this section the degree of closeness to this ideal is quantified.

Let

$$\phi = \text{force per unit length that filament exerts on surface} = \tau\kappa$$

where $\tau$ is the scalar tension in the filament and $\kappa$ is the vector curvature of the filament.

Note that the curvature vector can be resolved into a component tangent to the surface and a component normal to the surface $\kappa = \kappa_g + \kappa_n$ [2] where the tangent component of $\kappa$ is called the geodesic curvature vector and the normal component is called the normal curvature vector.

The force that a small length of filament exerts on the surface is

$$\phi ds = \tau\kappa ds = (\kappa_g+\kappa_n)\tau ds = \kappa_g\tau ds + \kappa_n\tau ds$$

Now, in order to avoid slippage of this small section of filament, the ratio of the magnitude of the tangent force to the magnitude of the normal force should be less than $\mu$, the coefficient of friction

$$\sigma = \frac{\|\kappa_g\| \tau ds}{\|\kappa_n\| \tau ds} = \left| \frac{k_g}{k_n} \right| < \mu$$

(or more precisely, $0 \leq \sigma = \frac{-|k_g|}{k_n} < \mu$ , since we want $k_n < 0$). We call this ratio of geodesic to normal curvature the slippage function $\sigma$. This function measures how close a given path comes to the ideal geodesic path ($\sigma = 0$). The evaluation of $\sigma$ is now detailed. First, if P is on the surface,

$$dP = P_x dx + P_\theta d\theta$$

and

$$ds^2 = dP \cdot dP = P_x \cdot P_x dx^2 + 2P_x \cdot P_\theta dx d\theta + P_\theta \cdot P_\theta d\theta^2$$

$$= E dx^2 + 2F dx d\theta + G d\theta^2$$

Hence,

$$(s')^2 = E + 2F\theta' + G\theta'^2$$

for a point on a path on the surface.

We now define a Cartan frame [2-4] relative to the surface, i.e., an orthonormal basis having two vectors tangent to the surface and a third normal to it. Let

$$e_1 = P_x / \|P_x\|$$

$$= P_x / (P_x \cdot P_x)^{\frac{1}{2}}$$

$$= P_x / E^{\frac{1}{2}}$$

and

$$e_3 = P_x x P_\theta / \|P_x x P_\theta\| = \nu / \|\nu\|$$

but from vector algebra,

$$(AxB) \cdot (CxD) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$$

Hence,

$$(P_x x P_\theta) \cdot (P_x x P_\theta) = \|P_x x P_\theta\|^2 = \|\nu\|^2$$

$$= (P_x \cdot P_x)(P_\theta \cdot P_\theta) - (P_x \cdot P_\theta)^2$$

$$= EG - F^2$$

Therefore,

$$e_3 = P_x x P_\theta/(EG-F^2)^{1/2}$$

Now

$$e_2 = e_3 x e_1$$

$$= (P_x x P_\theta) x P_x/(E^{1/2}(EG-F^2)^{1/2})$$

but again from vector algebra,

$$(AxB)xC = B(A \cdot C) - A(B \cdot C)$$

Hence,

$$(P_x x P_\theta) x P_x = P_\theta(P_x \cdot P_x) - P_x(P_x \cdot P_\theta)$$

$$= EP_\theta - FP_x$$

and

$$e_2 = (EP_\theta - FP_x)/(E^{1/2}(EG-F^2)^{1/2})$$

Now, $e_1$ and $e_2$ span the tangent space (plane) at any point; therefore, the path tangent vector t can be written

$$t = Ae_1 + Be_2$$

If t makes an angle $\omega$ with $e_1$,

$$e_1 \cdot t = A = \cos\omega$$

and

$$e_2 \cdot t = B = \sin\omega$$

Hence

$$t = e_1 \cos\omega + e_2 \sin\omega$$

where $\omega$ is the angle between the path and a meridian ($\theta$ = constant). Now

$$\kappa = \frac{dt}{ds} = \frac{de_1}{ds}\cos\omega + \frac{de_2}{ds}\sin\omega + (e_2\cos\omega - e_1\sin\omega)\frac{d\omega}{ds}$$

Let

$$T = e_3 x t$$

$$= e_3 x(e_1 \cos\omega + e_2 \sin\omega)$$

$$= e_2 \cos\omega - e_1 \sin\omega$$

Therefore,

$$\kappa = \frac{de_1}{ds} \cos \omega + \frac{de_2}{ds} \sin \omega + T \frac{d\omega}{ds}$$

and

$$T \cdot \kappa = T \cdot \frac{de_1}{ds} \cos \omega + T \cdot \frac{de_2}{ds} \sin \omega + \frac{d\omega}{ds}$$

but since

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1$$

one has

$$e_1 \cdot \frac{de_1}{ds} = e_2 \cdot \frac{de_2}{ds} = 0$$

and since

$$e_1 \cdot e_2 = 0$$

one has

$$\frac{de_1}{ds} \cdot e_2 = - \frac{de_2}{ds} \cdot e_1$$

We therefore have

$$T \cdot \frac{de_1}{ds} = e_2 \cdot \frac{de_1}{ds} \cos \omega$$

and

$$T \cdot \frac{de_2}{ds} = -e_1 \cdot \frac{de_2}{ds} \sin \omega = e_2 \cdot \frac{de_1}{ds} \sin \omega$$

Therefore

$$T \cdot \kappa = e_2 \cdot \frac{de_1}{ds} \cos^2 \omega + e_2 \cdot \frac{de_1}{ds} \sin^2 \omega + \frac{d\omega}{ds} = e_2 \cdot \frac{de_1}{ds} + \frac{d\omega}{ds}$$

but

$$\kappa = \kappa_g + \kappa_n \quad \text{and } T \cdot \kappa_n = 0$$

hence,

$$T \cdot \kappa = T \cdot \kappa_g = \|\kappa_g\| = k_g$$

and the geodesic curvature is

$$k_g = e_2 \cdot \frac{de_1}{ds} + \frac{d\omega}{ds} = (e_2 \cdot e_1{}' + \omega')/s'$$

Now

$$\sigma = \left| \frac{k_g}{k_n} \right|$$

so $k_n$ must be computed, but most of the work has already been done to find $k_n$.

$$k_n = e_3 \cdot \kappa = \frac{\nu}{\|\nu\|} \cdot \kappa = \frac{\nu \cdot P''}{\|\nu\|(s')^2} = \frac{\lambda}{\|\nu\|(s')^2}$$

Therefore,

$$\sigma = \left| \frac{e_2 \cdot e_1' + \omega'}{s'} \cdot \frac{\|\nu\|(s')^2}{\lambda} \right|$$

$$= \left| \frac{s'\|\nu\|}{\lambda}(e_2 \cdot e_1' + \omega') \right|$$

At this point, a few dot products must be computed. We have the identity

$$(P_u \cdot P_v)_w = P_v \cdot P_{uw} + P_u \cdot P_{vw}$$

Letting v = u,

$$P_u \cdot P_{uw} = \tfrac{1}{2}(P_u \cdot P_u)_w$$

Therefore

$$P_x \cdot P_{x\theta} = \tfrac{1}{2}E_\theta$$

$$P_\theta \cdot P_{\theta x} = \tfrac{1}{2}G_x$$

$$P_x \cdot P_{xx} = \tfrac{1}{2}E_x$$

and

$$P_\theta \cdot P_{\theta\theta} = \tfrac{1}{2}G_\theta$$

Letting w = u,

$$(P_u \cdot P_v)_u = P_v \cdot P_{uu} + P_u \cdot P_{uv}$$

$$= P_v \cdot P_{uu} + \tfrac{1}{2}(P_u \cdot P_u)_v$$

or

$$P_v \cdot P_{uu} = (P_u \cdot P_v)_u - \tfrac{1}{2}(P_u \cdot P_u)_v$$

Hence

$$P_x \cdot P_{\theta\theta} = F_\theta - \tfrac{1}{2}G_x$$

and

$$P_\theta \cdot P_{xx} = F_x - \tfrac{1}{2}E_\theta$$

Recalling

$$e_1 = P_x E^{-\frac{1}{2}}$$

and applying d/dx,

$$e_1' = P_x' E^{-\frac{1}{2}} + P_x (E^{-\frac{1}{2}})'$$

$$= (P_{xx} + P_{x\theta}\theta')E^{-\frac{1}{2}} + P_x (E^{-\frac{1}{2}})'$$

but

$$e_2 \cdot P_x = 0$$

Therefore

$$e_2 \cdot e_1' = (e_2 \cdot P_{xx} + e_2 \cdot P_{x\theta}\theta')/E^{\frac{1}{2}}$$

Now,

$$e_2 \cdot P_{xx} = (EP_\theta - FP_x) \cdot P_{xx}/(E^{\frac{1}{2}}\|\nu\|)$$

$$= (E(F_x - \tfrac{1}{2}E_\theta) - F(\tfrac{1}{2}E_x))/(E^{\frac{1}{2}}\|\nu\|)$$

$$e_2 \cdot P_{x\theta} = (EP_\theta - FP_x) \cdot P_{x\theta}/(E^{\frac{1}{2}}\|\nu\|)$$

$$= (E(\tfrac{1}{2}G_x) - F(\tfrac{1}{2}E_\theta))/(E^{\frac{1}{2}}\|\nu\|)$$

and therefore

$$e_2 \cdot e_1' = \{E(2F_x - E_\theta) - FE_x + (EG_x - FE_\theta)\theta'\}/(2E\|\nu\|)$$

Now, consider the meridian angle $\omega$

$$e_1 \cdot t = \cos\omega$$

$$= P_x \cdot t/E^{\frac{1}{2}}$$

$$= P_x \cdot \frac{dP}{ds}/E^{\frac{1}{2}}$$

$$= P_x \cdot P'/(s'E^{\frac{1}{2}})$$

$$= P_x \cdot (P_x + P_\theta \theta')/(s' E^{1/2})$$

$$= (E + F\theta')/(s' E^{1/2})$$

Therefore,

$$\text{Tan}\ \omega = \frac{(E(s')^2 - (E+F\theta')^2)^{1/2}}{E + F\theta'}$$

$$= \frac{(E(E+2F\theta'+G\theta'^2) - (E^2+2EF\theta'+F^2\theta'^2))^{1/2}}{E + F\theta'}$$

$$= \frac{(EG-F^2)^{1/2}\theta'}{E + F\theta'} = \frac{\|\nu\|\theta'}{E + F\theta'}$$

Solving for $\theta'$, we have

$$\theta' = \frac{E\ \text{Tan}\ \omega}{(EG-F^2)^{1/2} - F\ \text{Tan}\ \omega}$$

Therefore, $\sigma$ can be computed in terms of x, $\theta$, $\omega$, and $\omega'$.

The basic metric coefficients in terms of our parameterization are now computed:

$$P = iX + jY + kZ$$

but

$$P_x = i + jY_x + kZ_x$$

and

$$Y_x = Yr_x/r$$

$$Z_x = Zr_x/r$$

hence,

$$E = P_x \cdot P_x = 1 + Y_x^2 + Z_x^2 = 1 + \frac{r_x^2}{r^2}(Y^2+Z^2) = 1 + r_x^2$$

Now

$$P_\theta = jY_\theta + kZ_\theta$$

and

$$Y_\theta = Yr_\theta/r + Z$$

$$Z_\theta = Zr_\theta/r - Y$$

Hence

$$F = P_x \cdot P_\theta = Y_x Y_\theta + Z_x Z_\theta$$

$$= \frac{Yr_x}{r}\left(\frac{Yr_\theta}{r} + Z\right) + \frac{Zr_x}{r}\left(\frac{Zr_\theta}{r} - Y\right)$$

$$= \frac{r_x r_\theta}{r^2}(Y^2+Z^2)$$

$$= r_x r_\theta$$

Also

$$G = P_\theta \cdot P_\theta = Y_\theta^2 + Z_\theta^2$$

$$= \frac{Y^2 r_\theta^2}{r^2} + \frac{2YZr_\theta}{r} + Z^2$$

$$+ \frac{Z^2 r_\theta^2}{r^2} - \frac{2YZr_\theta}{r} + Y^2$$

$$= r^2 + r_\theta^2$$

Now some of the more important relations can be summarized:

$$\lambda = rr_{xx} + 2(rr_{x\theta} - r_x r_\theta)\theta' + (rr_{\theta\theta} - 2r_\theta^2 - r^2)(\theta')^2$$

$$(s')^2 = E + 2F\theta' + G(\theta')^2$$

$$\|\nu\|^2 = EG - F^2$$

$$\sigma = \left| \frac{s'\|\nu\|}{\lambda} (e_2 \cdot e_1' + \omega') \right|$$

$$e_2 \cdot e_1' = \{E(2F_x - E_\theta) - FE_x$$

$$+ (EG_x - FE_\theta)\theta'\}/(2E\|\nu\|)$$

$$\theta' = \frac{E \operatorname{Tan} \omega}{\|\nu\| - F \operatorname{Tan} \omega}$$

$$E = 1 + r_x^2$$

$$F = r_x r_\theta$$

$$G = r^2 + r_\theta^2$$

Now consider the two special cases addressed before. First, the general cylinder ($r_x = 0$):

$$E = 1 \ , \ F = 0 \ , \ G = r^2 + r_\theta^2 \ , \ G_x = 0 = E_x = E_\theta$$

$$\|\nu\| = G^{1/2}$$

$$e_2 \cdot e_1' = 0$$

$$\theta' = \frac{\operatorname{Tan}\omega}{G^{1/2}}$$

$$s' = (1+\operatorname{Tan}^2\omega)^{1/2} = \operatorname{Sec}\omega$$

$$\lambda = (r r_{\theta\theta} - 2r_\theta^2 - r^2)\ \frac{\operatorname{Tan}^2\omega}{G}$$

$$\sigma = \left| \frac{\operatorname{Sec}\omega\ G^{1/2}\ \omega'}{(r r_{\theta\theta} - 2r_\theta^2 - r^2)\ \frac{\operatorname{Tan}^2\omega}{G}} \right|$$

$$= \left| \frac{(r^2+r_\theta^2)^{3/2}\operatorname{Csc}\omega\ \operatorname{Cot}\omega\ \omega'}{(r r_{\theta\theta} - 2r_\theta^2 - r^2)} \right| = \left| \frac{(r^2+r_\theta^2)^{3/2}}{r r_{\theta\theta} - 2r_\theta^2 - r^2} \cdot \frac{d}{dx}\ (-\operatorname{Csc}\omega) \right|$$

hence,

$$\sigma = \frac{(r^2+r_\theta^2)^{3/2} |\ u'\ |}{r^2 + 2r_\theta^2 - r r_{\theta\theta}}$$

where $u = \operatorname{Csc}\omega$. Note that $u = 1$ at turning points and $u > 1$ between turning points. (A turning point is defined as a point at which $\theta' = \infty$ or $i \cdot t = 0$.) Also note that at points for which $u' = 0$, the path is geodesic. In addition, $\sigma \to 0$ if $r_{\theta\theta} \to -\infty$, while $r_\theta$ and $u'$ are bounded. This can be called a "knife edge" condition where $\sigma$ is zero due to infinite normal curvature instead of zero geodesic curvature. Now, consider the general surface of revolution ($r_\theta = 0$):

$$E = 1 + r'^2 \ , \ F = 0 \ , \ G = r^2 \ , \ E_\theta = 0 = G_\theta$$

$$\|\nu\| = (EG)^{1/2} = r\sqrt{1 + r'^2}$$

$$\theta' = \frac{E\ \operatorname{Tan}\omega}{\|\nu\|} = \left(\frac{E}{G}\right)^{1/2} \operatorname{Tan}\omega$$

$$e_2 \cdot e_1' = EG'\theta'/(2E\|\nu\|)$$

$$= \frac{G'}{2(EG)^{1/2}} \cdot \left(\frac{E}{G}\right)^{1/2} \operatorname{Tan}\omega$$

$$= \frac{G'}{2G} \operatorname{Tan}\omega$$

$$(s')^2 = E + G \cdot \frac{E}{G} \operatorname{Tan}^2 \omega$$

$$s' = E^{1/2} \operatorname{Sec} \omega = (1+r'^2)^{1/2} \operatorname{Sec} \omega$$

$$\lambda = rr'' - r^2(\theta')^2 = rr'' - r^2 \cdot \frac{E}{G} \operatorname{Tan}^2 \omega = rr'' - (1+r'^2) \operatorname{Tan}^2 \omega$$

$$\sigma = \left| \frac{E^{1/2}\operatorname{Sec} \omega \, (EG)^{1/2}}{rr'' - E \operatorname{Tan}^2 \omega} \left(\frac{G'}{2G} \operatorname{Tan} \omega + \omega'\right) \right|$$

$$= \left| \frac{r(1+r'^2)\operatorname{Sec} \omega \operatorname{Tan} \omega}{rr'' - (1+r'^2) \operatorname{Tan}^2 \omega} \left(\frac{r'}{r} + \operatorname{Cot} \omega \, \omega'\right) \right|$$

Now,

$$\frac{r'}{r} + \operatorname{Cot} \omega \, \omega' = \frac{d}{dx} \ln r + \frac{d}{dx} \ln \operatorname{Sin} \omega = \frac{d}{dx} \ln(r \operatorname{Sin} \omega)$$

It is clear that if $r \operatorname{Sin} \omega$ = constant, $\sigma$ will be zero and the path will be geodesic (Clairaut).

One can define a quasi-geodesic path on a surface of revolution by replacing Clairaut's relation [2,3,5] with $r \operatorname{Sin} \omega = r_0$ [1] where the function $r_0$ has the following properties:

- $r_0(x) = r(x)$ at exactly two values of x (turning points), and
- $r_0(x) < r(x)$ at all points between the turning points.

The function $r_0$ is called the polar radius function, because it is the radius of the surface at the boundaries of the uncovered polar regions [1]. Now, $\omega$ will be eliminated in favor of $r_0$. Since

$$\operatorname{Sin} \omega = \frac{r_0}{r}$$

one has that

$$\operatorname{Sec} \omega = \frac{r}{(r^2-r_0^2)^{1/2}}$$

$$\operatorname{Tan} \omega = \frac{r_0}{(r^2-r_0^2)^{1/2}}$$

$$\operatorname{Sec} \omega \operatorname{Tan} \omega = \frac{rr_0}{r^2 - r_0^2}$$

and

$$\frac{d}{dx} \ln r_o = \frac{r_o'}{r_o}$$

Hence, one finds after simplification that

$$\sigma = \frac{r^2 |r_o'|}{r_o^2 - rr''\left(\frac{r^2 - r_o^2}{1+r'^2}\right)}$$

Note the following: $\sigma = |r_o'|$ at turning points; $r_o' = 0$ at geodesic points; $\sigma = (\frac{r}{r_o})^2 |r_o'|$ if r is linear; $\sigma \to 0$ if $r'' \to -\infty$; while $r'$ and $r_o'$ are bounded (knife edge); and positivity of the denominator in $\sigma$ implies that

$$r_o^2 > \frac{r^3 r''}{1+r'^2+rr''}$$

It has been shown how the slippage function $\sigma$ can be computed for a general closed surface ($r_x \neq 0 \neq r_\theta$) and what simplifications take place in the $F \equiv 0$ cases. It should be emphasized, however, that $\sigma$ is more than just a number to be compared with the coefficient of friction $\mu$ to determine whether or not slippage occurs. The slippage function $\sigma$ measures pointwise path quality and should ultimately be usable to synthesize or define quality wrappable paths on general closed surfaces.

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